Coupled order-parameter system on a scale-free network

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The system of two scalar order parameters on a complex scale-free network is analyzed in the spirit of Landau theory. To add a microscopic background to the phenomenological approach, we also study a particular spin Hamiltonian that leads to coupled scalar order behavior using the mean-field approximation. Our results show that the system is characterized by either of two types of ordering: either one of the two order parameters is zero or both are nonzero but have the same value. While the critical exponents do not differ from those of a model with a single order parameter on a scale-free network, there are notable differences for the amplitude ratios and the susceptibilities. Another peculiarity of the model is that the transverse susceptibility is divergent at all $T < T_c$, when $O(n)$ symmetry is present. This behavior is related to the appearance of Goldstone modes.

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I. INTRODUCTION

The topology of many natural and manmade networks (social networks, biological, technological, and transportation systems) strongly differs from the topology of regular lattices or even random graphs. Often these networks show scale-free behavior [1–3]—the probability of a randomly chosen node to have a degree k (to have k links) follows a power law

$$P(k) = Ak^{-\lambda}. \tag{1}$$

Other integral parts of many real networks are a small-world effect and high clustering, resulting in specific features of cooperative phenomena on such systems. This has sparked interest in the analysis of different spin models on complex networks [4]. Such models have interesting applications. For example, the opinion of each individual of a social network may be represented by an Ising spin, e.g., for simple YES or NO alternatives. Such a model may describe phenomena of opinion formation, namely, individuals change their mind under the influence of their acquaintances [5]. Other more physical examples are given by assemblies of nanoscale particles with internal degrees of freedom, e.g., a spin [6]. These structures may assemble in different geometrical or fractal forms; therefore, spin models on scale-free networks can be used to mimic integrated nanosystems with nontrivial architecture [7].

The properties of the order-disorder phase transition of the Ising model on complex networks strongly depend on the node degree distribution (1). Numerical simulations [8] and analytical calculations [9] of the Ising model on Barabasi-Albert scale-free networks ($\lambda=3$) as well as different analytical approaches [10,11] and Monte Carlo simulations [12] for the Ising model on networks with arbitrary degree distributions have been performed. Three types of behavior were found depending on the respective behavior of the moments $\langle k^2 \rangle$ and $\langle k^4 \rangle$ of the degree distribution, which are related to the value of the λ exponent. Namely, if $\langle k^4 \rangle$ and $\langle k^2 \rangle$ are finite ($\lambda > 5$), the behavior of the system is described by the standard mean-field critical exponents. If $\langle k^4 \rangle$ diverges and $\langle k^2 \rangle$ is finite, which corresponds to $3 < \lambda \le 5$, the critical behavior is governed by either mean-field exponents with logarithmic corrections ($\lambda=5$) or by nontrivial λ-dependent critical exponents. Finally, if both $\langle k^4 \rangle$ and $\langle k^2 \rangle$ diverge ($2 < \lambda \le 3$), the critical temperature becomes divergent (for infinite-size networks), and the system is always ordered. Furthermore, other models on scale-free networks, namely, the *XY* [13] and the Potts [14,15] models (for a more detailed list, see, e.g., [4,16]) also show peculiarities depending on the value of λ.

Rather recently, critical phenomena on complex networks have been studied in the spirit of Landau theory [17]. The power of the latter is that it is independent of the origin of the interactions between the particles, and therefore it may be applied to a wide range of systems. The main feature of the phenomenological theory of critical phenomena on complex networks that differs from standard Landau theory is the dependence of the coefficients on the moments $\langle k^i \rangle$ of the degree distribution (1).

Landau theory for two interacting scalar order parameters is widely used to analyze systems with several possible types of ordering (e.g., ferromagnetic and antiferromagnetic, ferroelectric and ferromagnetic, or structural and magnetic ordering). Such combinations of order parameters may be described by a model of two scalar order parameters x_1, x_2, which are coupled [18,19]. Assuming that the Landau free energy is analytical and symmetric with respect to the signs of x_1 and x_2 the lowest order coupling is biquadratic,

$$\Phi(\vec{x}, T) = \frac{a}{2}(T - T_c)|\vec{x}|^2 + \frac{b}{4}|\vec{x}|^4 + \frac{c}{4}x_1^2 x_2^2, \tag{2}$$

where $\vec{x} = (x_1, x_2)$; $|\vec{x}|^2 = x_1^2 + x_2^2$; a, b, c are the phenomenological Landau parameters; and T and T_c are the temperature and

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the critical temperature, correspondingly. A possible application of this model of coupled order parameters to a social network may reflect the coupling between the preferences for a candidate and a party in an election (or similar scenarios) [20]. The free energy (2) corresponds to the free energy of an n-vector anisotropic cubic model in the case $n=2$. The latter is obtained from the $O(n)$ invariant free energy by adding invariants of the symmetry group B_n of the n-dimensional hypercube [21].

The aim of our work is to generalize the Landau theory for models on complex networks [17] to the case of two interacting order parameters with a free-energy symmetry given by Eq. (2). The structure of our paper is as follows. The next section (Sec. II) lays out the basic assumptions of the theory and the peculiarities of the free-energy construction and compares the approach with a microscopic model. Section III describes the stable states and the phase diagrams of the system. The behavior of the thermodynamic functions, the isothermal susceptibilities, and the heat capacity is described in Sec. IV. We conclude with an outlook in Sec. V. Some details of our calculations are given in Appendixes A and B.

II. FREE ENERGY

This section is devoted to the construction of a generalized Landau theory for a system with two coupled order parameters on a network (Sec. II A). Besides, we derive a corresponding free energy starting with a microscopic spin Hamiltonian and compare both approaches (Sec. II B).

A. Generalized Landau theory

In the spirit of the Landau approach we assume that the system may display some ordering which can be quantitatively characterized by two order parameters x_1 and x_2. For convenience let us introduce a vector $\vec{x}=(x_1,x_2)$. Following the work of Ref. [17], we assume that the Landau free energy per site is not only a function of the order parameters, the conjugated field $\vec{h}$, and the temperature but also depends on the node degree distribution $P(k)$,

$$\Phi(\vec{x},T,\vec{h}) = \int_1^{k_{max}} dk\, P(k)f(\vec{x},k\vec{x}) - \vec{h}\vec{x}, \qquad (3)$$

where $f(\vec{x},k\vec{x})$ represents the contribution to the free energy of an individual node of degree k and k_{max} is the maximal node degree of the network. Note that $k_{max}\rightarrow\infty$ is implied for an infinite-size system with a power-law node degree distribution as in Eq. (1). That $f(\vec{x},k\vec{x})$ depends not only on the order parameters x_1 and x_2 but also on $k\vec{x}$ may be understood by simple reasoning. It reflects that any node with k neighbors is subjected to a field $k\vec{x}$ of these neighbors.

The next basic assumption in the case of a scalar order parameter x is that $f(x,kx)$ is an analytical function of x and kx [17]. In the case of two order parameters, we assume that $f(\vec{x},k\vec{x})$ is now an analytical function of x_1, x_2, kx_1, and kx_2 and may be represented as a series in their powers,

$$f(\vec{x},k\vec{x}) = \sum_{l_1,l_2,m_1,m_2=0}^{\infty} f_{l_1 l_2 m_1 m_2} x_1^{l_1} x_2^{l_2} (kx_1)^{m_1} (kx_2)^{m_2}, \qquad (4)$$

where $f_{l_1 l_2 m_1 m_2}$ are functions which in general may depend on the temperature T and an external field $\vec{h}$. Moreover, some relations between these coefficients are implied by the symmetry of the system as described by Eq. (2). In this case the function $f(\vec{x},k\vec{x})$ may be represented as

$$f(\vec{x},k\vec{x}) = f_0 + \sum_{i=0}^{2} a_i k^i |\vec{x}|^2 + \sum_{i=0}^{4} b_i k^i |\vec{x}|^4 + \sum_{i=0}^{4} c_i k^i \sum_{\mu=1}^{2} x_\mu^4 + \cdots, \qquad (5)$$

where f_0, a_i, b_i, c_i are convenient notations for the coefficients $f_{l_1 l_2 m_1 m_2}$.

Naturally, the free energy (3) must be finite if the order parameters are finite. This condition is satisfied in particular if the behavior of the function $f(\vec{x},k\vec{x})$ at large $k|\vec{x}|\rightarrow\infty$ is bounded by

$$f(\vec{x},k\vec{x}) \sim k|\vec{x}|, \quad k|\vec{x}|\rightarrow\infty. \qquad (6)$$

Assumptions (3) and (5) as well as condition (6) serve as our basis to analyze the phase transitions in the coupled order-parameter system following the standard approach of Landau theory [22].

Substituting Eq. (5) into Eq. (3) and taking into account that the coefficient of $|\vec{x}|^2$ in the free energy is equal to zero at the critical point, the equation for the critical temperature T_c as a function of the moments of the degree distribution is found in the same manner as in the case of a scalar order parameter [17] to be

$$a_0(T_c) + a_1(T_c)\langle k\rangle + a_2(T_c)\langle k^2\rangle = 0. \qquad (7)$$

If $a_0(T_c)=0$, the critical temperature is a function of $\langle k^2\rangle/\langle k\rangle$. This statement is in accordance with the exact result for the Ising model on networks obtained analytically [10,11] and confirmed numerically [12], where T_c follows

$$\frac{1}{T_c} = \frac{1}{2}\ln\left(\frac{\langle k^2\rangle}{\langle k^2\rangle - 2\langle k\rangle}\right). \qquad (8)$$

Before we embark to calculate the free energy, let us discuss an essential point that is the origin of many of the peculiarities of cooperative phenomena on networks. For scale-free networks with a node degree distribution as in Eq. (1) one finds in general that all moments $\langle k^i\rangle$ with $i<\lambda-1$ are finite, whereas all moments with $i\geq\lambda-1$ diverge. If we restrict the series in Eq. (5) to the fourth power of the order parameter, there are no relevant divergent moments for $\lambda>5$. Nevertheless, if $\langle k^4\rangle$ or lower moments of the degree distribution are divergent ($\lambda\leq 5$), as often found for real networks, the free energy (3) at first sight may seem to be infinite for any nonzero values of the order parameters, a behavior which is certainly unphysical. In fact, the correct way to calculate the free energy is to take into account all the orders of series (5). This procedure ensures a behavior of the function $f(\vec{x},k\vec{x})$ at large values of $k|\vec{x}|\rightarrow\infty$ as described by Eq.

(6). Therefore, we collect all terms in Eq. (5) containing k^i with $i \geq \lambda - 1$ together with the highest orders of series (5) in a function $g(\vec{x}, k\vec{x})$ as follows:

$$f(\vec{x}, k\vec{x}) = f_0 + \sum_{i=0}^{i_2} a_i k^i |\vec{x}|^2 + \sum_{i=0}^{i_4} b_i k^i |\vec{x}|^4 + \sum_{i=0}^{i_4} c_i k^i \sum_{\mu=1}^{2} x_\mu^4$$
$$+ g(\vec{x}, k\vec{x}). \qquad (9)$$

Here i_2 is the maximal integer that satisfies both conditions $i_2 \leq 2$ and $i_2 < \lambda - 1$. Respectively, i_4 is the maximal integer that satisfies both $i_4 \leq 4$ and $i_4 < \lambda - 1$. Now it is straightforward to integrate the part of $f(\vec{x}, k\vec{x})$ that does not include $g(\vec{x}, k\vec{x})$. Any peculiarities are connected with the integration of $g(\vec{x}, k\vec{x})$. Let us therefore investigate the properties of this function. Comparing Eq. (9) with Eq. (5) one finds that for small values of $k|\vec{x}|$ and for $3 < \lambda \leq 5$ this function behaves as

$$g(\vec{x}, k\vec{x}) = b_4 (k|\vec{x}|)^4 + c_4 \frac{x_1^4 + x_2^4}{|\vec{x}|^4} (k|\vec{x}|)^4, \quad k|\vec{x}| \to 0. \quad (10)$$

For $2 < \lambda \leq 3$ one finds the following behavior:

$$g(\vec{x}, k\vec{x}) = a_2 (k|\vec{x}|)^2 + \left(b_2 + c_2 \frac{x_1^4 + x_2^4}{|\vec{x}|^4} \right) |\vec{x}|^2 (k|\vec{x}|)^2$$
$$+ \left(b_3 + c_3 \frac{x_1^4 + x_2^4}{|\vec{x}|^4} \right) |\vec{x}| (k|\vec{x}|)^3$$
$$+ \left(b_4 + c_4 \frac{x_1^4 + x_2^4}{|\vec{x}|^4} \right) (k|\vec{x}|)^4, \quad k|\vec{x}| \to 0. \quad (11)$$

We do not consider the case $\lambda \leq 2$ here as far as then $\langle k \rangle$ is not defined.

In order to satisfy condition (6) for a finite free energy, the behavior of $g(\vec{x}, k\vec{x})$ is restricted for large values of $k|\vec{x}| \to \infty$ by the highest explicitly written term of $f(\vec{x}, k\vec{x})$ in Eq. (9). Namely, for $k|\vec{x}| \to \infty$, $g(\vec{x}, k\vec{x})$ is restricted by

$$g(\vec{x}, k\vec{x}) \sim \begin{cases} (k|\vec{x}|)^3, & 4 < \lambda \leq 5 \\ (k|\vec{x}|)^2, & 3 < \lambda \leq 4 \\ (k|\vec{x}|), & 2 < \lambda \leq 3. \end{cases} \quad (12)$$

To perform the integration of $g(\vec{x}, k\vec{x})$ in Eq. (3), note that it actually depends on $\vec{x}$ and $k|\vec{x}|$, $g(\vec{x}, k\vec{x}) \equiv g(\vec{x}, k|\vec{x}|)$ [see Eqs. (10)–(12)]. Let us pass to a new variable $y = k|\vec{x}|$, which ranges from $|\vec{x}|$ to infinity for infinite-size networks. For a network with a power-law node degree distribution (1), one may then write

$$\int_1^\infty dk \, P(k) g(\vec{x}, k|\vec{x}|) = A|\vec{x}|^{\lambda-1} \int_{|\vec{x}|}^\infty \frac{dy}{y^\lambda} g(\vec{x}, y). \quad (13)$$

As only the asymptotics of $g(\vec{x}, y)$ are fixed, let us write

$$\int_{|\vec{x}|}^\infty \frac{dy}{y^\lambda} g(\vec{x}, y) = \int_\varepsilon^\infty \frac{dy}{y^\lambda} g(\vec{x}, y) - \int_\varepsilon^{|\vec{x}|} \frac{dy}{y^\lambda} g(\vec{x}, y), \quad (14)$$

where ε is a small positive number $0 < \varepsilon < |\vec{x}|$; obviously, both sides of this expression do not depend on ε. The first term on its right-hand side is convergent, due to the asymptotic behavior (12). In the second term $g(\vec{x}, y)$ may be

replaced by its expansion for small values of y [Eqs. (10) and (11)].

Following this procedure, one may obtain the free energy. Details of the integration of expression (14) for the case $4 < \lambda < 5$ are given in Appendix A (analog calculations can be performed for other values of λ). In the following, we will consider zero external magnetic field $\vec{h} = 0$ and drop the explicit $\vec{h}$ dependence from our notations. Let us present the resulting expressions for the Landau free energy for different ranges of values of λ. We treat the cases (a) $\lambda > 5$, (b) $\lambda = 5$, (c) $3 < \lambda < 5$, (d) $\lambda = 3$, and (e) $2 < \lambda < 3$. As we will see, differences between the usual Landau theory and that on a scale-free network become apparent starting from the marginal case $\lambda = 5$.

(a) Case $\lambda > 5$. In this case the free energy may be found easily by substituting Eq. (5) into Eq. (3) and performing the integration. The free energy reads

$$\Phi(\vec{x}, T) = f_0 + \frac{a}{2}(T - T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^4 + \frac{c^{(\lambda)}}{4} x_1^2 x_2^2. \quad (15)$$

The specific network properties are expressed by the coefficients

$$\frac{a}{2}(T - T_c) = a_1 \langle k \rangle + a_2 \langle k^2 \rangle \quad (16)$$

$$b^{(\lambda)} = 4b_4 \langle k^4 \rangle, \quad c^{(\lambda)} = 4c_4 \langle k^4 \rangle. \quad (17)$$

As seen below, Eq. (16) also holds for $3 < \lambda \leq 5$.

(b) Case $\lambda = 5$. In this case the free energy reads

$$\Phi(\vec{x}, T) = f_0 + \frac{a}{2}(T - T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^4 \ln\frac{1}{|\vec{x}|} + \frac{c^{(\lambda)}}{4} x_1^2 x_2^2 \ln\frac{1}{|\vec{x}|}. \quad (18)$$

In this marginal case the free energy displays logarithmic corrections to the standard mean-field behavior. The coefficient $a(T - T_c)$ is described by Eq. (16) and the other coefficients are as follows:

$$b^{(\lambda)} = 4A(b_4 + c_4), \quad c^{(\lambda)} = -8Ac_4. \quad (19)$$

(c) Case $3 < \lambda < 5$. Here, the free energy reads

$$\Phi(\vec{x}, T) = f_0 + \frac{a}{2}(T - T_c)|\vec{x}|^2 + \frac{b^{(\lambda)}}{4}|\vec{x}|^{\lambda-1} + \frac{c^{(\lambda)}}{4}\frac{x_1^2 x_2^2}{|\vec{x}|^4}|\vec{x}|^{\lambda-1}. \quad (20)$$

In this case the free energy (20) explicitly depends on λ. The coefficient $a(T - T_c)$ is also described by Eq. (16), whereas to get expressions for $b^{(\lambda)}$ and $c^{(\lambda)}$ from the integration of $g(\vec{x}, k\vec{x})$, one needs to perform explicit calculations in parallel to those presented in Appendix A.

(d) Case $\lambda = 3$. Here, the free energy reads

$$\Phi(\vec{x}, T) = f_0 + C|\vec{x}|^2 - D|\vec{x}|^2 \ln\frac{1}{|\vec{x}|} + E\frac{x_1^2 x_2^2}{|\vec{x}|^4}|\vec{x}|^2. \quad (21)$$

(e) Case $2 < \lambda < 3$. In this case we find a free energy of the form

$$\Phi(\vec{x}, T) = f_0 + C'|\vec{x}|^2 + D'|\vec{x}|^{\lambda-1} + E'\frac{x_1^2 x_2^2}{|\vec{x}|^4}|\vec{x}|^{\lambda-1}. \quad (22)$$

For cases (d) and (e) we give explicitly only the expressions for D (for $\lambda = 3$) and C' (for $2 < \lambda < 3$) as

$$D = -Aa_2, \quad C' = a_1\langle k \rangle - \frac{Aa_2}{3-\lambda}. \quad (23)$$

For an example of how to calculate the other coefficients by integration of $g(\vec{x}, k\vec{x})$, see Appendix A.

Note, that for $2 < \lambda \le 3$, the term of order $|\vec{x}|^2$ is no more the leading one. Terms of lower order of magnitude become relevant. In particular there is a term $|\vec{x}|^2 \ln|\vec{x}|^{-1}$ for $\lambda = 3$ and a term $|\vec{x}|^{\lambda-1}$ for $2 < \lambda < 3$.

Before passing to the details of the phase diagram that results from the expressions for the Landau free energies (15), (18), and (20)–(22), we first proceed to show that the Landau free energy may also be derived from a spin system on a network by calculating its partition function in the simplest of approximations.

B. Anisotropic Hamiltonian

One of the ways to get the Landau free energy with two coupled scalar order parameters is to start with two coupled spin subsystems [23]. Another way is to consider a single spin system with a cubic anisotropy term. Let us use the second option, considering a spin model on a complex network described by a Hamiltonian with an anisotropic term

$$H = -J\sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j + u\sum_{i=1}^{N}\sum_{\nu=1}^{2} s_{\nu,i}^4, \quad (24)$$

where $\vec{s}_i$ and $\vec{s}_j$ are spins on nodes i and j correspondingly, J and u are the coupling and the anisotropy constants, the notation $\Sigma_{\langle i,j \rangle}$ denotes the summation over all pairs of connected nodes, the index ν numbers the components of the two-component vector, and $\vec{s}_i \cdot \vec{s}_j = \Sigma_{\nu=1}^{2} s_{\nu,i} s_{\nu,j}$ is a scalar product. Again, as above we will consider the case when the network node degree distribution obeys a power-law decay (1). Note, that Hamiltonian (24) represents an n-vector anisotropic cubic model [21] in the case $n = 2$.

Here, we consider Hamiltonian (24) in the spirit of a mean-field approach. Applying the mean-field approach to a model that is defined on a regular lattice (equal degree k for all nodes), each node is characterized by the same mean spin $\langle \vec{s} \rangle$ and experiences the effective field $\langle k \rangle \langle \vec{s} \rangle$ of its $\langle k \rangle$ neighbors. In the case of a complex network, this assumption may be applied only to nodes with the same degree: in the simplest approximation each k-degree node experiences the same mean spin $\langle \vec{s} \rangle_k$. In return, the mean spin value per node $\langle \vec{s} \rangle$ may be expressed in terms of $\langle \vec{s} \rangle_k$ as

$$\langle \vec{s} \rangle = \sum_k P(k)\langle \vec{s} \rangle_k. \quad (25)$$

On the other hand, it can be found from the thermodynamical definition

$$\langle \vec{s} \rangle = -\left(\frac{\partial G(T, \vec{h})}{\partial \vec{h}}\right)_T. \quad (26)$$

Here $G(T, \vec{h})$ is the appropriate thermodynamical potential and $\vec{h}$ is an external field.

Node i experiences the effective field of its k_i neighbors. This field may be quantitatively characterized by the mean value $\vec{\sigma}^i$ of the spins surrounding the ith node [10,11],

$$\vec{\sigma}^i = \frac{1}{k_i}\sum_{\langle j \rangle} \vec{s}_j. \quad (27)$$

Here the sum over j spans the k_i nearest neighbors of node i. Now, in the spirit of the mean-field theory one assumes that $\vec{\sigma}^i$ does not depend on the node number i,

$$\vec{\sigma}^i = \vec{\sigma}, \quad i = 1, \dots, N. \quad (28)$$

Note, that the above defined value $\vec{\sigma}$ differs from the mean spin value per node $\langle \vec{s} \rangle$. Equation (26) gives us the relation between the mean spin $\langle \vec{s} \rangle$ and the effective spin $\vec{\sigma}$ per neighbor.

To proceed with Hamiltonian (24), we introduce the deviation of every spin component $s_{\nu,i}$ from the corresponding component of the average spin per neighbor $\vec{\sigma}$,

$$\Delta s_{\nu,i} = s_{\nu,i} - \sigma_\nu. \quad (29)$$

Substituting $s_{\nu,i} = \sigma_\nu + \Delta s_{\nu,i}$ into the scalar product in Eq. (24) and neglecting the terms of order $O((\Delta s)^2)$ we arrive at the mean-field Hamiltonian,

$$H_{MF} = \sum_{i=1}^{N} H_{MF}^i \quad (30)$$

with

$$H_{MF}^i = \frac{1}{2}J\langle k \rangle \sigma^2 - Jk_i\sum_{\nu=1}^{2} \sigma_\nu s_{\nu,i} + u\sum_{\nu=1}^{2} s_{\nu,i}^4. \quad (31)$$

Here, $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Now, the partition function is reduced to a product of single-site traces,

$$Z_{MF} = \prod_{i=1}^{N} \text{Tr}_i\, e^{-H_{MF}^i/T}. \quad (32)$$

Here, the single-site trace $\text{Tr}_i(\cdots)$ denotes the integration over all possible directions of $\vec{s}_i$,

$$\text{Tr}_i(\cdots) = \int d\vec{s}_i\, \delta(L - |\vec{s}_i|)(\cdots). \quad (33)$$

The δ function ensures that all spins $\vec{s}_i$ have the same absolute value L. Substituting Eq. (31) into Eq. (32) and taking the trace (some details of the calculations are given in Appendix B) one arrives at the free energy per site,

$$F(\vec{\sigma}, T) = -T/N \ln Z_{MF}. \quad (34)$$

As usual in the mean-field approach, the free energy (34) depends on the macroscopic mean-field variable $\vec{\sigma}$. The expression for the free energy per site reads

$$F(\vec{\sigma},T) = \frac{1}{N}\sum_{i=1}^{N} \hat{f}(\vec{\sigma}, k_i\vec{\sigma}) \qquad (35)$$

with

$$\hat{f}(\vec{\sigma}, k_i\vec{\sigma}) = -T\ln(2\pi L) + \frac{1}{2}\frac{T^2}{JL^2}k_i|\vec{\xi}|^2$$
$$- T\ln\left(I_0(k_i|\vec{\xi}|) - \frac{uL^4}{T}\left[6\frac{I_2(k_i|\vec{\xi}|)}{(k_i|\vec{\xi}|)^2} + 6\frac{I_3(k_i|\vec{\xi}|)}{k_i|\vec{\xi}|} \right.\right.$$
$$\left.\left. + I_4(k_i|\vec{\xi}|)\sum_{\nu=1}^{2} \xi_\nu^4/|\vec{\xi}|^4 \right] \right) \qquad (36)$$

and

$$\vec{\xi} = \frac{JL}{T}\vec{\sigma}. \qquad (37)$$

In Eq. (36), $I_n(z)$ are modified Bessel functions [24] of the first kind,

$$I_n(z) = \frac{1}{2\pi i}\oint e^{(z/2)(\omega+1/\omega)}\omega^{-n-1}d\omega. \qquad (38)$$

It is instructive to observe that in Eq. (36) the function $\hat{f}$ depends both on $k\vec{\sigma}$ and on $\vec{\sigma}$ [via the second term in Eq. (36)]—a property postulated in the Landau approach (see Sec. II A).

We now replace the sum over nodes in Eq. (35) with a sum over node degrees,

$$F(\vec{\sigma},T) = \frac{1}{N}\sum_{i=1}^{N} \hat{f}(\vec{\sigma}, k_i\vec{\sigma}) = \sum_{k=1}^{k_{max}} P(k)\hat{f}(\vec{\sigma}, k\vec{\sigma}). \qquad (39)$$

Here $P(k)$ is the density of nodes with degree k (1) and $\hat{f}(\vec{\sigma}, k\vec{\sigma})$ represents the contribution of a single k-degree node. Note that $\hat{f}(\vec{\sigma}, k\vec{\sigma})$ actually depends on $\vec{\sigma}$ and $k\sigma$,

$$\hat{f}(\vec{\sigma}, k\vec{\sigma}) \equiv \hat{f}(\vec{\sigma}, k\sigma). \qquad (40)$$

Therefore, we further replace the sum over k in Eq. (39) with an integral over k (3) and introduce $y=k\sigma$ as the variable of integration,

$$F(\vec{\sigma},T) = \sigma^{\lambda-1}\int_\sigma^\infty P(y)\hat{f}(\vec{\sigma},y)dy. \qquad (41)$$

The convergence of integral (41) for large y can be derived from the asymptotic behavior of the function $I_\nu(z)$ as follows[24]:

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \quad z\to\infty. \qquad (42)$$

Namely, substituting Eq. (42) into Eq. (36), one finally arrives at

$$\hat{f}(\vec{\sigma},y) \sim y, \quad y\to\infty. \qquad (43)$$

The last estimate, together with the power-law behavior (1) proves the convergence of the expression for free energy (35) for $\lambda > 2$.

The behavior of $\hat{f}(\vec{\sigma},y)$ for small $y\to 0$ and a small anisotropy parameter $u/T\ll 1$ is characterized by the smallest term of the Bessel function expansions [24] in

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{q=0}^\infty \frac{(z^2/4)^q}{k!\,\Gamma(\nu+q+1)}, \qquad (44)$$

where $\Gamma(\rho)$ is the Euler gamma function. Now, substituting Eq. (44) into Eq. (36) one arrives at

$$\hat{f}(\vec{\sigma},y) = f_0 + \frac{1}{2}J\sigma y - \frac{1}{4}\frac{(JL)^2}{T}y^2 + \frac{1}{64}\left(1 - \frac{49}{8}\frac{uL^4}{T}\right)\frac{(JL)^4}{T^3}y^4$$
$$+ \frac{1}{384}uL^4\frac{(JL)^4}{T^4}\frac{\sigma_1^4+\sigma_2^4}{\sigma^4}y^4 + \cdots \qquad (45)$$

with

$$f_0 = -T\ln(2\pi L) + \tfrac{3}{4}uL^4. \qquad (46)$$

We will perform the integration in Eq. (41) using the expansion of $\hat{f}(\vec{\sigma},y)$ (45) and its asymptotics (43) at $y\to\infty$.

Those terms of expansion (45) that are well behaved with respect to the integration in Eq. (41) may be easily integrated. These are the terms, in which y^μ appears with $\mu < \lambda - 1$. The integration of the remainder of series (45) [let us denote it as $\hat{g}(\vec{\sigma},y)$] needs some special care. Using the asymptotic behavior of $\hat{g}(\vec{\sigma},y)$ at small and large values of y, the integration is to be performed in the same way as for $g(\vec{x},y)$ (see Sec. II A) to obtain the free energy as described above.

To complete the calculations we now pass from the average spin $\vec{\sigma}$ per nearest neighbor to the mean spin $\langle\vec{s}\rangle$ of a node. Solving Eq. (26) for $\langle\vec{s}\rangle$ one finds in a linear approximation in $\vec{\sigma}$ and u,

$$\langle\vec{s}\rangle = \frac{J\langle k\rangle L^2}{2T}\vec{\sigma}. \qquad (47)$$

Substituting Eq. (47) into Eq. (45) one finally obtains the free-energy density as

$$f(\langle\vec{s}\rangle, k\langle\vec{s}\rangle) = f_0 + \frac{2T^2}{J\langle k\rangle^2 L^4}k\langle\vec{s}\rangle^2 - \frac{T}{\langle k\rangle^2 L^2}k^2\langle\vec{s}\rangle^2$$
$$+ \frac{T}{4\langle k\rangle^4 L^4}\left(1 - \frac{49}{8}\frac{uL^4}{T}\right)k^4\langle\vec{s}\rangle^4 + \frac{uk^4}{24\langle k\rangle^4}(\langle s_1\rangle^4$$
$$+ \langle s_2\rangle^4) + \cdots. \qquad (48)$$

Note that taking into account higher-order corrections in Eq. (47) does not change the free energy at the critical point.

Expression (48) serves as an example for a microscopic interpretation of the phenomenological Landau free energy $\Phi(\vec{x},T)$ [Eqs. (3) and (5)]. Indeed, the two-component order parameter $\vec{x}$ in Eq. (5) may be interpreted as the two-component mean spin (magnetization) per site $\langle\vec{s}\rangle$ in Eq. (48). The remaining phenomenological Landau parameters

may be found by direct comparison of expressions (5) and (48). In this way, the value of f_0 in Eq. (5) has a microscopic representation in terms of Eq. (46), while the coefficients a_i read

$$a_0 = 0, \quad a_1 = \frac{2T^2}{J\langle k \rangle^2 L^4}, \quad a_2 = -\frac{T}{\langle k \rangle^2 L^2}. \quad (49)$$

Recall that in the frames of the Landau approach the assumption $a_0 = 0$ leads to the dependence of T_c on $\langle k^2 \rangle / \langle k \rangle$. Now we find the expression for T_c in the microscopic model as

$$T_c = \frac{JL^2}{2} \frac{\langle k^2 \rangle}{\langle k \rangle}. \quad (50)$$

The values for the other coefficients in the Landau expansion are as follows:

$$b_i = c_i = 0, \quad i = 0, \dots, 3, \quad (51)$$

$$b_4 = \frac{T}{4\langle k \rangle^4 L^4}\left(1 - \frac{49}{8}\frac{uL^4}{T}\right), \quad c_4 = \frac{u}{24\langle k \rangle^4}. \quad (52)$$

Moreover, the microscopic approach allows us to estimate the temperature dependence of the proportionality coefficients in the free-energy expressions. The latter is of primary importance for the case $2 < \lambda \le 3$, when the critical temperature diverges. Then

$$C, C' \sim T^2, \quad D, D' \sim T, \quad E, E' \sim T^0. \quad (53)$$

In the following, we pass to a more detailed analysis of the Landau free energy (3).

III. PHASE DIAGRAMS

Having determined the behavior of free energy (3), let us investigate the stable states of the system. The latter may be found from the minimization of the free energy. The condition of stationarity requires the first derivatives of the free energy to vanish,

$$\frac{\partial \Phi(\vec{x}, T)}{\partial x_1} = 0, \quad \frac{\partial \Phi(\vec{x}, T)}{\partial x_2} = 0. \quad (54)$$

The stationary point is a minimum if both eigenvalues of the matrix of second derivatives

$$\omega_{\mu\nu} = \frac{\partial^2 \Phi(\vec{x}, T)}{\partial x_\mu \, \partial x_\nu}, \quad \mu, \nu = 1, 2 \quad (55)$$

are positive. This condition may also be written as

$$\mathrm{Re}(\omega_{\mu\mu}) > 0, \quad \det(\omega_{\mu\nu}) > 0, \quad \mu, \nu = 1, 2. \quad (56)$$

From a physical point of view, the minimum of the free energy requires positive isothermal susceptibilities. In the following we consider the stable states of the system with coupled order parameters for the relevant ranges of the exponent λ, discussed for the generalized Landau free energy.

A. Case $\lambda > 5$

For $\lambda > 5$ the system is described by the Landau free energy (15), whereas the type of the ordering below T_c depends
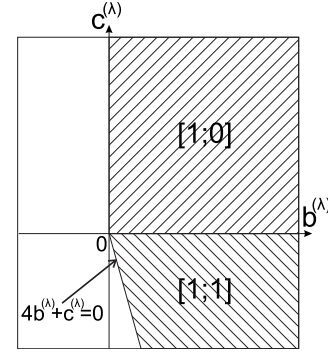

FIG. 1. The phase diagram for the coupled two-component order-parameter model (2) on a complex scale-free network. The picture shows what type of order is realized in the different phases depending on the free-energy parameters $b^{(\lambda)}$ and $c^{(\lambda)}$. The blank part of the phase diagram corresponds to absence of a stable phase. An ordered phase exists only if $b^{(\lambda)} > 0$. The sign of the coefficient $c^{(\lambda)}$ separates two phases. Namely, positive values of $c^{(\lambda)} > 0$ correspond to phases with only one nonzero order-parameter component ([1,0] or [0,1]); negative values $c^{(\lambda)} < 0$ that satisfy the condition $4b^{(\lambda)} + c^{(\lambda)} > 0$ correspond to the ordered phase [1,1], where both order parameters have the same nonzero value.

on the interplay between the fourth-order couplings. If $c^{(\lambda)} > 0$ and $b^{(\lambda)} > 0$, the system is characterized by order-parameter components

$$x_1 = \sqrt{\frac{a}{b^{(\lambda)}}}(T_c - T)^\beta, \quad x_2 = 0. \quad (57)$$

If $c^{(\lambda)} < 0$ and $4b^{(\lambda)} + c^{(\lambda)} > 0$, both order parameters have the same value

$$x_1 = x_2 = \sqrt{\frac{2a}{4b^{(\lambda)} + c^{(\lambda)}}}(T_c - T)^\beta, \quad (58)$$

with $\beta = 1/2$. Here and below we do not write explicitly one more solution $x_1 = 0$, $x_2 \ne 0$ which is symmetric to Eq. (57) and which is stable under the same conditions as the solution $x_1 \ne 0$, $x_2 = 0$. The resulting phase diagram is shown in Fig. 1. The blank parts of the phase diagram correspond to cases where no stable state exists. For these values of a, $b^{(\lambda)}$, and $c^{(\lambda)}$ the condition of stability of the thermodynamic potential cannot be satisfied [i.e., the asymptotics $\Phi(\vec{x}, T) \to \infty$ for $|\vec{x}| \to \infty$ do not hold]; therefore, the system is undefined for this range of parameters.

B. Case $\lambda = 5$

If the exponent λ is at its marginal value $\lambda = 5$, the free energy is described by Eq. (18). For temperatures below T_c stable states exist only if $b^{(\lambda)} > 0$. For $c^{(\lambda)} > 0$, the system is described near the critical point $T \to T_c$ by an ordered phase with

$$x_1 \approx \sqrt{\frac{2a}{b^{(\lambda)}}}\frac{(T_c - T)^\beta}{[\ln(T_c - T)^{-1}]^{1/2}}, \quad x_2 = 0. \quad (59)$$

If $-4b^{(\lambda)} < c^{(\lambda)} < 0$, the ordered phase at $T \to T_c$ is characterized by the order parameters

$$x_1 = x_2 \approx 2\sqrt{\frac{a}{4b^{(\lambda)} + c^{(\lambda)}}} \frac{(T_c - T)^\beta}{[\ln(T_c - T)^{-1}]^{1/2}}. \qquad (60)$$

Note that expressions (59) and (60) represent approximate solutions of Eq. (54), which is transcendental when considering the free energy (18). Both phases are characterized by the same value of the critical exponent $\beta = 1/2$. For other values of $b^{(\lambda)}$ and $c^{(\lambda)}$, the free energy (18) does not lead to any equilibrium stable state, as in the case $\lambda > 5$. These results are also depicted in the phase diagram in Fig. 1.

C. Case $3 < \lambda < 5$

For degree distributions governed by an exponent in the range $3 < \lambda < 5$, the system is described by the free energy (20). Below T_c stable states exist only if $b^{(\lambda)} > 0$. Namely, there are two stable phases: one with

$$x_1 = \left(\frac{4a}{(\lambda - 1)b^{(\lambda)}}\right)^{1/(\lambda-3)} (T_c - T)^\beta, \quad x_2 = 0 \qquad (61)$$

and a second one with

$$x_1 = x_2 = \frac{1}{\sqrt{2}}\left(\frac{16a}{(\lambda - 1)(4b^{(\lambda)} + c^{(\lambda)})}\right)^{1/(\lambda-3)} (T_c - T)^\beta, \qquad (62)$$

where $\beta = \frac{1}{\lambda - 3}$. The regions where these states are realized are shown in the phase diagram Fig. 1. If $c^{(\lambda)} > 0$ the system is in the stable state (61). Otherwise for negative $c^{(\lambda)} < 0$ and $4b^{(\lambda)} + c^{(\lambda)} > 0$, the system is described by the order-parameter components (62). As observed earlier for larger values of λ, the stability conditions (54) and (56) cannot be satisfied for other values of $b^{(\lambda)}$ and $c^{(\lambda)}$ in the free energy (20).

We conclude that if $\langle k^4 \rangle$ diverges but $\langle k^2 \rangle$ is finite ($3 < \lambda \le 5$), the critical behavior differs from the classical mean-field behavior. Furthermore, if $\lambda = 5$, logarithmic corrections appear, and if $3 < \lambda < 5$ the critical exponents are functions of λ. Note that for all values of $\lambda > 3$ considered above there exists a finite critical temperature. This will not be the case for the values of λ considered below.

D. Case $2 < \lambda \le 3$

When the exponent λ is in the range $2 < \lambda \le 3$ and the second moment $\langle k^2 \rangle$ of the node degree distribution (1) diverges, one may infer from Eq. (7) that the order-disorder phase transition does not occur at any finite temperature. Taking into account that at $T = 0$ the system is ordered, the system keeps order at any finite temperature, as has been confirmed for the Ising model on the infinite-size Barabasi-Albert scale-free network [8,9].

In the case $\lambda = 3$, the free energy is given by Eq. (21). If the parameter D is positive as follows from Eqs. (1), (23), and (49), the system is always ordered. The type of order found depends on the parameter E. If E is positive only one order parameter has nonzero value,

$$x_1 = e^{-(2C+D)/2D}, \quad x_2 = 0. \qquad (63)$$

For negative values of E both order parameters are nonzero and have equal values

$$x_1 = x_2 = \frac{1}{\sqrt{2}} e^{-(4C+2D+E)/4D}. \qquad (64)$$

The high-temperature dependence of both order parameters in view of Eq. (53) follows

$$x_1, x_2 \sim e^{-\eta T} \qquad (65)$$

with some constant η determined by the coefficients of the high-temperature behavior of C and D in Eq. (53).

In the case $2 < \lambda < 3$ the system is described by the free energy (22). Assuming the anisotropy parameter to be small ($E' \ll D'$), one finds stable states of the system. If $C' > 0$ [corresponding to Eqs. (1), (23), and (49)] and $D' < 0$, the system is always ordered and the type of order depends on the sign of the anisotropy parameter E'. Namely, if $E' > 0$, the ordered phase is characterized by

$$x_1 = \left(\frac{2}{\lambda - 1}\right)^{1/(\lambda-3)}\left(-\frac{C'}{D'}\right)^{1/(\lambda-3)}, \quad x_2 = 0. \qquad (66)$$

If $E' < 0$, both order parameters are nonzero with

$$x_1 = x_2 = \left(\frac{2^{(9-\lambda)/2}}{\lambda - 1}\right)^{1/(\lambda-3)}\left(-\frac{C'}{4D' + E'}\right)^{1/(\lambda-3)}. \qquad (67)$$

Taking into account the high-temperature dependence of C' and D' (53), the temperature dependencies of the nonzero order parameters for $T \to \infty$ can be found as

$$x_1, x_2 \sim T^{-1/(3-\lambda)}. \qquad (68)$$

This corresponds to the scalar theory results [10,11]. As one may expect, for all $2 < \lambda \le 3$ both x_1 and x_2 vanish only at infinitely large temperature.

IV. REACTION OF THE SYSTEM TO AN EXTERNAL ACTION

A. Isothermal susceptibilities

In the case of two order parameters, the behavior of the system in an external field is described by two quantities. The longitudinal susceptibility $\chi_\parallel$ describes the reaction of the system to an external field applied along the order-parameter direction. In turn, $\chi_\perp$ describes the reaction to a transverse external field.

In the disordered state and in the absence of an external field the system is isotropic and therefore there is no difference between $\chi_\parallel$ and $\chi_\perp$. In the general case the susceptibility matrix $\chi_{\mu\nu} = (\partial x_\mu / \partial h_\nu)|_T$ (see, e.g., [25])

$$\chi_{\mu\nu} = \delta_{\mu\nu}\chi_\parallel + (1 - \delta_{\mu\nu})\chi_\perp, \quad \mu, \nu = 1, 2 \qquad (69)$$

depends on both $\chi_\parallel$ and $\chi_\perp$, which may be found as the inverse eigenvalues of the matrix of second-order derivatives of the free energy (55). Here $\delta_{\mu\nu}$ is the Kronecker symbol.

Thus, above the critical temperature $T > T_c$ both susceptibilities have the same dependence, for all values of $\lambda > 3$,

$$\chi_\parallel = \chi_\perp = \frac{1}{a}(T - T_c)^{-\gamma} \tag{70}$$

with the standard mean-field critical exponent $\gamma = 1$. As mentioned above, there is no disordered state in a scale-free network of infinite size with $2 < \lambda \leq 3$.

For all $\lambda > 3$ and below the critical temperature both $\chi_\parallel$ and $\chi_\perp$ follow a power law with the mean-field critical exponent $\gamma = 1$, as also found for the case of a scalar order parameter. Furthermore, the absolute value of the longitudinal susceptibility $\chi_\parallel$ coincides with the susceptibility χ found for the scalar case [17]. Our results are

$$\chi_\parallel = \begin{cases} \dfrac{1}{2a}(T_c - T)^{-\gamma}, & \lambda > 5 \\[2mm] \dfrac{1}{(\lambda - 3)a}(T_c - T)^{-\gamma}, & 3 < \lambda \leq 5. \end{cases} \tag{71}$$

The absolute value of the transverse susceptibility $\chi_\perp$ depends on both λ and the type of order. So, *for* $\lambda > 5$

$$\chi_\perp = \begin{cases} \dfrac{2b^{(\lambda)}}{ac^{(\lambda)}}(T_c - T)^{-\gamma} & \text{for } \vec{x} = [1,0] \\[3mm] -\dfrac{4b^{(\lambda)} + c^{(\lambda)}}{2ac^{(\lambda)}}(T_c - T)^{-\gamma} & \text{for } \vec{x} = [1,1]. \end{cases} \tag{72}$$

As one may see from Eq. (72), when the coefficient $c^{(\lambda)} = 0$, and thus the system described by the free energy (2) becomes isotropic [26], the transverse susceptibility diverges $\chi_\perp \to \infty$ for any $T < T_c$. This behavior of $\chi_\perp$ is quite physical and is a consequence of the free-energy symmetry: an infinitely small external field applied in a direction perpendicular to the order parameter immediately changes the order-parameter orientation. This is the Goldstone phenomenon, corresponding to the existence of a soft excitation mode in the ordered phase [25].

For $3 < \lambda \leq 5$, the transverse susceptibility is given by

$$\chi_\perp = \begin{cases} \dfrac{(\lambda - 1)b^{(\lambda)}}{2ac^{(\lambda)}}(T_c - T)^{-\gamma} & \text{for } \vec{x} = [1,0] \\[3mm] -\dfrac{(\lambda - 1)(4b^{(\lambda)} + c^{(\lambda)})}{8ac^{(\lambda)}}(T_c - T)^{-\gamma} & \text{for } \vec{x} = [1,1]. \end{cases} \tag{73}$$

As discussed above for the case $\lambda > 5$, again the transverse susceptibility diverges for a vanishing parameter $c^{(\lambda)} = 0$.

For $\lambda = 3$ the longitudinal susceptibility reads

$$\chi_\parallel = \frac{1}{2D} \sim T^{-1}. \tag{74}$$

The sign of the transverse susceptibilities depends on the phase,

$$\chi_\perp = \begin{cases} 1/2E, & \vec{x} = [1,0] \\ -1/2E, & \vec{x} = [1,1]. \end{cases} \tag{75}$$

For $2 < \lambda < 3$ the behavior is similar. The longitudinal susceptibility follows

$$\chi_\parallel = \frac{1}{2(3 - \lambda)C'} \sim T^{-2}, \tag{76}$$

while we have different transverse susceptibilities in the two stable phases,

$$\chi_\perp = \begin{cases} -\dfrac{\lambda - 1}{4}\dfrac{D'}{C'E'}, & \vec{x} = [1,0] \\[3mm] \dfrac{\lambda - 1}{16}\dfrac{4D' + E'}{C'E'}, & \vec{x} = [1,1]. \end{cases} \tag{77}$$

As we learn from the above equations (70)–(73), the singularity at the critical point is governed by the mean-field value of the critical exponent $\gamma = 1$. This reproduces the behavior observed within the Landau theory for systems with a scalar order parameter on scale-free networks [17]. In this respect passing to a system with a more complicated symmetry does not appear to modify the universal critical exponents. Note however, the significant change in other universal quantities, namely, the susceptibility amplitude ratios. Defining the amplitudes for the susceptibilities by

$$\chi_i = \begin{cases} \Gamma_{+,i}(T - T_c)^{-\gamma}, & T > T_c \\ \Gamma_{-,i}(T_c - T)^{-\gamma}, & T < T_c, \quad i = \parallel, \perp, \end{cases} \tag{78}$$

let us compare the amplitude ratios Γ_+/Γ_- for longitudinal and transverse susceptibilities for different phases. Recall that for a scalar order-parameter, Landau theory gives $\Gamma_+/\Gamma_- = 2$ [22]. Correspondingly, for the free energy (2) one finds for the longitudinal susceptibility

$$(\Gamma_+/\Gamma_-)_\parallel = 2 \tag{79}$$

while the amplitude ratio for the transverse susceptibility depends on the type of the ordered phase,

$$(\Gamma_+/\Gamma_-)_\perp = \begin{cases} c/2b, & \vec{x} = [1,0] \\ -2c/(4b + c), & \vec{x} = [1,1], \end{cases} \tag{80}$$

where the notations $[1,0]$ and $[1,1]$ indicate the corresponding phases. As one can see, the amplitude ratios (80) depend on the couplings b, c. For $\lambda > 5$ the free energy (15) is equivalent to that of Eq. (2) however with coefficients b and c given by Eq. (17). Thus the ratio Γ_+/Γ_- attains the same values as for the systems with the free energy (2). For $\lambda \leq 5$ the ratio Γ_+/Γ_- is a function of λ, similar as it holds for the order-parameter critical exponent β. So, the amplitude ratio for the longitudinal susceptibility for all $3 < \lambda \leq 5$ reads

$$(\Gamma_+/\Gamma_-)_\parallel = (\lambda - 3). \tag{81}$$

For the transverse susceptibilities the ratio depends on the phase and, respectively, on the values of the coefficients of the free-energy function. Namely, the susceptibility ratios are

$$(\Gamma_+/\Gamma_-)_\perp = \begin{cases} \dfrac{2c^{(\lambda)}}{(\lambda - 1)b^{(\lambda)}}, & \vec{x} = [1,0] \\[3mm] -\dfrac{8c^{(\lambda)}}{(\lambda - 1)(4b^{(\lambda)} + c^{(\lambda)})}, & \vec{x} = [1,1]. \end{cases} \tag{82}$$

The amplitude ratios for the different ranges of λ and phases are summarized in Table I. Summarizing, we note that for all the range of $\lambda > 3$ (where the critical temperature T_c exists), the behavior of the system with respect to an external field is

TABLE I. Amplitude ratios for different ranges of the λ exponent. Second column: amplitude ratio for the longitudinal susceptibilities; third and fourth columns: amplitude ratio for the transverse susceptibilities for two different phases, denoted by [1,0] and [1,1], respectively.

λ	$(\Gamma_+/\Gamma_-)_\parallel$	$(\Gamma_+/\Gamma_-)_{\perp[1,0]}$	$(\Gamma_+/\Gamma_-)_{\perp[1,1]}$
$\lambda > 5$	2	$c^{(\lambda)}/2b^{(\lambda)}$	$-2c^{(\lambda)}/(4b^{(\lambda)}+c^{(\lambda)})$
$3 < \lambda \le 5$	$\lambda - 3$	$2c^{(\lambda)}/(\lambda-1)b^{(\lambda)}$	$-8c^{(\lambda)}/(\lambda-1)(4b^{(\lambda)}+c^{(\lambda)})$

governed by a mean-field critical exponent $\gamma = 1$, but the amplitude ratios have nontrivial forms.

B. Heat capacity

The heat capacity describes the behavior of the system with respect to a change in temperature,

$$c_h = T \left(\frac{dS}{dT} \right)_h. \tag{83}$$

In the frames of the Landau theory, the coefficient of $|\vec{x}|^2$ in the free energy changes its sign at the critical temperature. The other coefficients are assumed to be temperature independent. Note that for a phase transition on a scale-free network, this assumption holds also for $\lambda > 3$, whereas for $2 < \lambda \le 3$ the temperature dependencies of the coefficients are described by Eq. (53). Then one may find the entropy of the system as

$$S = - \left(\frac{\partial \Phi}{\partial T} \right)_x, \tag{84}$$

which for $\lambda > 3$ reduces to the simple expression

$$S = -\frac{a}{2} |\vec{x}|^2, \tag{85}$$

where $|\vec{x}|$ is a function of temperature and external field. Substituting stable solutions that follow from Eq. (54) into Eq. (84), one finds the entropy S at fixed external field $h=0$ for each phase. Respectively, the heat capacity may be found by taking the derivative of the entropy with respect to the temperature in Eq. (83).

It is known that for a second-order phase transition in simple magnets the Landau theory predicts a step in the heat capacity at the critical temperature T_c. The behavior of the heat capacity for a system on a scale-free network is richer. In the standard mean-field region $\lambda > 5$ and below T_c the heat capacity decreases linearly with the decrease in temperature. At the critical point the step in the heat capacity is

$$\delta c_h = \frac{a^2}{2b^{(\lambda)}} T_c. \tag{86}$$

Taking into account the microscopic relations (49), (50), and (52), one obtains the step in the heat capacity as follows:

$$\delta c_h = 2 \frac{\langle k^2 \rangle^2}{\langle k^4 \rangle}, \tag{87}$$

which vanishes for $\lambda \to 5$. For $3 < \lambda \le 5$ there is no step of c_h at the critical point. Namely, for $\lambda = 5$ we find the following expressions for the heat capacity at $T < T_c$ in the phases [1,0] and [1,1], correspondingly:

$$c_h \approx \frac{a^2}{b^{(\lambda)}} \frac{T}{\ln(T_c - T)^{-1}} \quad \text{in phase } [1,0], \tag{88}$$

$$c_h \approx \frac{4a^2}{4b^{(\lambda)} + c^{(\lambda)}} \frac{T}{\ln(T_c - T)^{-1}} \quad \text{in phase } [1,1]. \tag{89}$$

In the case $3 < \lambda < 5$ the corresponding formulas read

$$c_h = \frac{a}{\lambda - 3} \left[\frac{4a}{(\lambda-1)b^{(\lambda)}} \right]^{2/(\lambda-3)}$$
$$\times T(T_c - T)^{(5-\lambda)/(\lambda-3)} \quad \text{in phase } [1,0], \tag{90}$$

$$c_h = \frac{a}{\lambda - 3} \left[\frac{16a}{(\lambda-1)(4b^{(\lambda)} + c^{(\lambda)})} \right]^{2/(\lambda-3)}$$
$$\times T(T_c - T)^{(5-\lambda)/(\lambda-3)} \quad \text{in phase } [1,1]. \tag{91}$$

As one can see from Eqs. (88)–(91), the heat capacity vanishes as $T \to T_c$ which differs from the case $\lambda > 5$, where the corresponding value at T_c is given by Eq. (86). Nevertheless, a maximum of c_h is still present for $3 < \lambda \le 5$. Only now, it is shifted from T_c to the temperature region $T < T_c$. The low-temperature behavior of the heat capacity at $3 < \lambda \le 5$ resembles that for $\lambda > 5$: $c_h \sim T$. The heat capacity vanishes both at $T=0$ and $T=T_c$ and possesses a maximum at an intermediate temperature $0 < T_0 < T_c$. For $\lambda = 5$ this temperature coincides with T_c, whereas for lower values of λ we find

$$T_0 = \frac{\lambda - 3}{2} T_c, \quad 3 < \lambda < 5. \tag{92}$$

Taking into account the explicit calculations of Sec. II A,

$$T_0 = (\lambda - 2) \frac{JL^2}{4}, \quad 3 < \lambda < 5. \tag{93}$$

In Fig. 2 we show the typical behavior of c_h for different values of $3 < \lambda \le 5$. There, we represent Eqs. (90) and (91) in the form

$$c_h = c_0 \frac{T}{T_c} \left(1 - \frac{T}{T_c} \right)^{(5-\lambda)/(\lambda-3)} \tag{94}$$

and plot c_h/c_0 as a function of a scaled variable T/T_c.

As λ approaches from above 3, the critical temperature increases and becomes infinite for $2 < \lambda \le 3$: the system is always ordered and the type of the ordered phase is governed by signs of the coefficients E, E' in the Landau free energies (21) and (22). For both ordered phases we obtain that the high-temperature behavior of c_h is described by
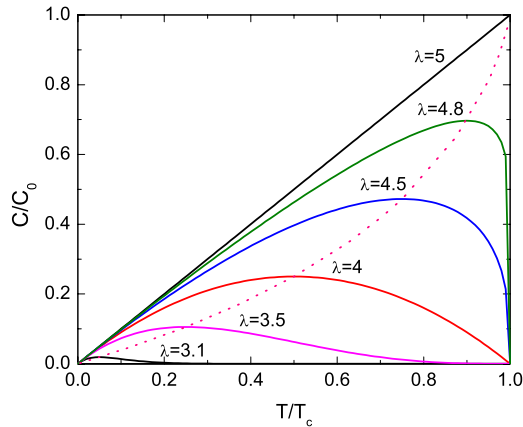
FIG. 2. (Color online) Typical behavior of the heat capacity for different values of λ in the range of $3 < \lambda \leq 5$. The dotted curve shows the position of the maximum at temperature T_0 [see Eq. (92)].

$$c_h \sim \begin{cases} T^2 e^{-\zeta T}, & \lambda = 3 \\ T^{-(\lambda-1)/(3-\lambda)}, & 2 < \lambda < 3, \end{cases} \quad (95)$$

where ζ depends on the coefficients defined in Eq. (53).

As we have observed for the order parameter and the susceptibility, the character of the temperature dependence of the heat capacity for the system of two coupled order parameters reproduces the one obtained for a single scalar order parameter [11,17]. Note however the different amplitudes for this behavior resulting from Eqs. (90) and (91).

V. CONCLUSIONS

Models that display phase transitions with two coupled order parameters serve as archetypes to describe the phase behavior in systems with several possible types of ordering [18,19]. For example, a system may display both ferromagnetic and antiferromagnetic orders with a coupling between the two. Others may show magnetic and superconducting, ferroelectric and ferromagnetic, or structural and magnetic order. These models are known for their rich phase diagrams and nontrivial critical behavior. Inspired by these observations, this paper investigates the phase transitions of a corresponding model defined on a scale-free network. Besides the academic interest, this problem may have implications for models of opinion formation on social networks when opinions on different issues may be coupled as, e.g., the preferences for both a candidate and a political party [20].

Our analysis was based on thermodynamic arguments in the spirit of Landau theory, as suited for the description of phase transitions on scale-free networks [17]. To add a microscopic background to the phenomenological approach we have also studied a particular spin Hamiltonian that leads to coupled scalar order behavior using the mean-field approximation. Our results show that for the scale-free networks with a degree distribution governed by an exponent $\lambda > 2$ the system is characterized by either of two types of ordering. Either one of the two order parameters is zero (the [1,0] or the [0,1] phase) or both are nonzero but have the same value

(the [1,1] phase). Along with the critical behavior of scalar order-parameter systems on scale-free networks, the order of the phase transition in the coupled scalar order-parameter system depends on the decay of the node-degree distribution. For rapidly decaying distributions ($\lambda \geq 5$) the second-order phase transition is similar to that described by usual Landau theory. Nevertheless unique features appear as λ decreases: whereas the magnetic susceptibility (and higher than second derivatives of the free energy with respect to the magnetic field) remains divergent at T_c for all $3 < \lambda < 5$, the order of the lowest divergent temperature derivative of the free energy depends on λ [27]. Namely, as seen from Eqs. (90) and (91), it is the third-order derivative for $4 < \lambda < 5$, the fourth order for $3\frac{2}{3} < \lambda < 5$, and so on until it is only the infinite-order derivative that diverges for $\lambda = 3$: the order of the phase transition becomes infinite [11,23].

The critical behavior of the model considered gives rise to nontrivial critical exponents, amplitude ratios, and susceptibilities. While the critical exponents do not differ from those of a model with a single order parameter on a scale-free network [17], there are notable differences for the amplitude ratios and susceptibilities. Another peculiarity of the model is that the transverse susceptibility is divergent at all $T < T_c$, when $O(n)$ symmetry is present. Such behavior is related to the appearance of Goldstone modes. It is worth to mention a peculiarity in the behavior of the specific heat. Whereas for $\lambda \geq 5$ it has a jump at the critical temperature T_c, this jump disappears for $\lambda < 5$. The heat capacity vanishes both at $T = 0$ and $T = T_c$ and possesses maximum at an intermediate temperature $0 < T_0 < T_c$.

The phenomena observed serve as evidence of a rich critical behavior caused by scale-free properties of the underlying network structure. An attractive feature for the theoretical analysis of this behavior is of course that nontrivial effects are found already in very simple approximations. Natural continuations of our study will include extensions beyond the mean-field approach, taking into account order-parameter fluctuations and, further, studies of dynamic processes and in particular the critical dynamics resulting at or near the critical point. Such studies need to be based on more detailed information about the structure of the network than the degree distribution, such as provided by the adjacency matrix or the network Laplacian (Kirchhoff matrix), e.g., in terms of their respective eigenvalue spectra [28,29].

APPENDIX A

In order to perform the integration in Eq. (14), assume $4 < \lambda < 5$. For other values of the exponent λ the calculations can be performed in a similar way. From Eqs. (10) and (12) we derive the following asymptotics of $g(\vec{x}, y)$:

$$g(\vec{x}, y) = \left(b_4 + c_4 \frac{x_1^4 + x_2^4}{|\vec{x}|^4} \right) y^4, \quad y \to 0, \quad (A1)$$

$$g(\vec{x}, y) \sim y^3, \quad y \to \infty. \tag{A2}$$

To analyze Eq. (14), let us define

$$Q_1(\varepsilon, \vec{x}, \lambda) = \int_\varepsilon^\infty \frac{dy}{y^\lambda} g(\vec{x}, y), \tag{A3}$$

$$Q_2(\varepsilon, \vec{x}, \lambda) = \int_\varepsilon^{|\vec{x}|} \frac{dy}{y^\lambda} g(\vec{x}, y). \tag{A4}$$

From the asymptotic behavior (A2) one may infer that Q_1 is convergent. Assuming that near the critical point the absolute value of the order parameter $|\vec{x}| \ll 1$ is small, we replace the function $g(\vec{x}, y)$ in Q_2 with its expansion (A1) at small values of y and perform the integration. Then one obtains

$$\int_{|\vec{x}|}^\infty \frac{dy}{y^\lambda} g(\vec{x}, y) = Q(\vec{x}, \lambda) - \left(b_4 + c_4 \frac{x_1^4 + x_2^4}{|\vec{x}|^4} \right) \frac{|\vec{x}|^{5-\lambda}}{5-\lambda}, \tag{A5}$$

where

$$Q(\vec{x}, \lambda) = Q_1(\varepsilon, \vec{x}, \lambda) + \left(b_4 + c_4 \frac{x_1^4 + x_2^4}{|\vec{x}|^4} \right) \frac{\varepsilon^{5-\lambda}}{5-\lambda}. \tag{A6}$$

Naturally, $Q(\vec{x}, \lambda)$ does not depend on ε [as $\int_{|\vec{x}|}^\infty \frac{dy}{y^\lambda} g(\vec{x}, y)$ does not depend on ε], while the dependence of $Q(\vec{x}, \lambda)$ on $\vec{x}$ is reasonable [see asymptotics (A1) and Eq. (A6)] to be assumed as follows:

$$Q(\vec{x}, \lambda) = v_1 + v_2 \frac{x_1^4 + x_2^4}{|\vec{x}|^4}, \tag{A7}$$

where v_1 and v_2 are some coefficients, which are in general dependent on λ and the temperature.

Substituting these results into Eq. (13), one obtains

$$\int_1^\infty dk \, P(k) g(\vec{x}, k|\vec{x}|) = A Q(\vec{x}, \lambda) |\vec{x}|^{\lambda-1}$$
$$+ \frac{A}{5-\lambda} \left(b_4 + c_4 \frac{x_1^4 + x_2^4}{|\vec{x}|^4} \right) |\vec{x}|^4. \tag{A8}$$

In the region of λ considered ($4 < \lambda < 5$) near the critical point the leading term includes a factor $|\vec{x}|^{\lambda-1}$ and correspondingly $Q(\vec{x}, \lambda)$ is part of the relevant terms of the free energy.

APPENDIX B

Here we calculate the partition function (32) with H_{MF}^i described by Eq. (31). To calculate

$$Z_{MF} = \prod_{i=1}^N Z_{MF}^i = \prod_{i=1}^N \text{Tr}_i \, e^{-H_{MF}^i/T}, \tag{B1}$$

we use the following property of the δ function:

$$\delta(L - |\vec{s}_i|) = 2L\delta(L^2 - |\vec{s}_i|^2) \tag{B2}$$

and use its Fourier presentation

$$\delta(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dv_0 \, e^{v_0 x}. \tag{B3}$$

Then Z_{MF}^i reads

$$Z_{MF}^i = \frac{Lz_i}{\pi i} \int_{-\infty}^\infty d\vec{s}_i \int_{-i\infty}^{i\infty} dv_0 \, e^{v_0 L^2} \prod_{\nu=1}^2 e^{-u_0 s_{\nu,i}^4 - v_0 s_{\nu,i}^2 + j_i \sigma_\nu s_{\nu,i}} \tag{B4}$$

with

$$z_i = e^{-Jk_i \sigma^2/2T}, \quad u_0 = \frac{u}{T}, \quad j_i = \frac{J}{T} k_i. \tag{B5}$$

Now, let us use the representation

$$\exp\{-u_0 s_{\nu,i}^4 - v_0 s_{\nu,i}^2 + j_i \sigma_\nu s_{\nu,i}\} = \exp\left\{ -\frac{u_0}{j_i^4} \frac{\partial^4}{\partial \sigma_\nu^4} \right\}$$
$$\times \exp\{-v_0 s_{\nu,i}^2 + j_i \sigma_\nu s_{\nu,i}\}, \tag{B6}$$

where $\exp(-\frac{u_0}{j_i^4} \frac{\partial^4}{\partial \sigma_\nu^4})$ is interpreted as

$$\exp\left\{ -\frac{u_0}{j_i^4} \frac{\partial^4}{\partial \sigma_\nu^4} \right\} = 1 - \frac{u_0}{j_i^4} \frac{\partial^4}{\partial \sigma_\nu^4} + \frac{1}{2!} \left(\frac{u_0}{j_i^4} \frac{\partial^4}{\partial \sigma_\nu^4} \right)^2 + \cdots. \tag{B7}$$

Substituting Eq. (B6) into Eq. (B4), one obtains

$$Z_{MF}^i = \frac{Lz_i}{\pi i} \left[\prod_{\nu=1}^2 \exp\left(-\frac{u_0}{j_i^4} \frac{\partial^4}{\partial \sigma_\nu^4} \right) \int_{-\infty}^\infty ds_{\nu,i} \right]$$
$$\times \int_{-i\infty}^{i\infty} dv_0 \, e^{v_0 L^2} \prod_{\nu=1}^2 \exp(-v_0 s_{\nu,i}^2 + j_i \sigma_\nu s_{\nu,i}). \tag{B8}$$

To change the order of integration over $S_{\nu,i}$ and v_0, we multiply the integrand with $\exp\{\alpha(L^2 - |\vec{s}_i|^2)\}$, which is equal to unity due to the constraint. Let us choose α to be sufficiently large to satisfy $(v_0 + \alpha)s_{\nu,i}^2 - j_i \sigma_\nu s_{\nu,i} > 0$. Then one may use the Poisson integral

$$\int_{-\infty}^\infty dx \, e^{-ax^2 + bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a} \tag{B9}$$

to obtain

$$Z_{MF}^i = \frac{Lz_i}{\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} dv \, e^{vL^2} \frac{\pi}{v} \prod_{\nu=1}^2 e^{-(u_0/j_i^4)(\partial^4/\partial \sigma_\nu^4)} e^{j_i^2 \sigma_i^2/4v}. \tag{B10}$$

Assuming the anisotropy parameter u to be small and, respectively, $u_0 \ll 1$ we keep only the term linear in u_0 in expansion (B10). Then Eq. (B10) may be written as

$$Z_{MF}^i = \frac{\pi L z_i}{\pi i} \int d\omega_i \exp\left[\frac{1}{2} j_i L\sigma(\omega_i + 1/\omega_i)\right]$$
$$\times \omega_i^{-1}\left\{1 - u_0 L^4\left[6\frac{\omega_i^{-2}}{(j_i L\sigma)^2} + 6\frac{\omega_i^{-3}}{j_i L\sigma} + \frac{\sigma_1^4 + \sigma_2^4}{\sigma^4}\omega_i^{-4}\right]\right\}, \tag{B11}$$

where

$$\omega_i = \frac{2Lv}{j_i\sigma}. \tag{B12}$$

The integration path for the variable ω_i in the integral in Eq. (B11) ranges from $2L\alpha/j_i\sigma - i\infty$ to $2L\alpha/j_i\sigma + i\infty$.

Using the definition of the modified Bessel function of the first kind,

$$I_n(z) = \frac{1}{2\pi i}\oint e^{(z/2)(\omega+1/\omega)}\omega^{-n-1}d\omega, \tag{B13}$$

one may write

$$Z_{MF}^i = 2\pi L z_i\left\{I_0(j_i L\sigma) - u_0 L^4\left[6\frac{I_2(j_i L\sigma)}{(j_i L\sigma)^2} + 6\frac{I_3(j_i L\sigma)}{j_i L\sigma}\right.\right.$$
$$\left.\left. + \frac{\sigma_1^4 + \sigma_2^4}{\sigma^4}I_4(j_i L\sigma)\right]\right\}. \tag{B14}$$

Substituting Eq. (B14) into Eq. (B1) one finally obtains the partition function and, respectively, one may find the free energy per site $F(\vec{\sigma}, T) = -T/N \ln Z_{MF}$. Again, keeping terms linear in u, one obtains free energy (35).

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